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INCLUSION-EXCLUSION INEQUALITIES

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If μ is a positive measure, and $A_1, \ldots, A_n$ are measurable sets, the sequences $S_0, \ldots, S_n$ and $P_{[0]}, \ldots, P_{[n]}$ are related by the inclusion-exclusion equalities. Inequalities among the S_i are based on the obvious $P_{[k]} \geqq 0$. Letting $M_k = S_k \Big/ \binom{n}{k}$ = the average measure of the intersection of k of the sets A_i, it is shown that $(-1)^k \Delta^k M_i \geqq 0$ for $i+k \leqq n$. The case $k=1$ yields Fréchet's inequalities, and $k=2$ yields Gumbel's and K. L. Chung's inequalities. Generalizations are given involving k-th order divided differences. Using convexity arguments, it is shown that for $S_0=1$, $S_N \geqq \binom{S_1}{N}$ when $S_1 \geqq N-1$, and $\binom{\nu}{k-1} S_N \geqq \binom{\nu}{N-1} S_k + \binom{\nu}{N}\binom{\nu}{k-1} - \binom{\nu}{N-1}\binom{\nu}{k}$ for $1 \leqq k < N \leqq n$ and $\nu = 0, 1, \ldots$. Asymptotic results as $n \to \infty$ are obtained. In particular it is shown that for fixed N, $\sum_{i=0}^{N} a_i M_i \geqq 0$ for all sequences $M_0, \ldots, M_n$ of sufficiently large length if and only if $\sum_{i=0}^{N} a_i t^i > 0$ for $0 < t < 1$.

Suppose X is a set and μ is a nonnegative, finitely additive measure on some Boolean algebra of subsets of X with $\mu(X) < \infty$. Let $A_1, \ldots, A_n$ be measurable sets. Now set

$$S_k = \sum_{i_1 < \ldots < i_k} \mu(A_{i_1} \cap \ldots \cap A_{i_k}), \quad 0 \leqq k \leqq n \tag{1}$$

$$P_{[k]} = \sum_{\substack{i_1 < \ldots < i_k \\ j_1 < \ldots < j_{n-k}}} \mu(A_{i_1} \cap \ldots \cap A_{i_k} \cap \bar{A}_{j_1} \cap \ldots \cap \bar{A}_{j_{n-k}}), \quad 0 \leqq k \leqq n. \tag{2}$$

By convention, $S_0 = \mu(X)$. The well known inclusion-exclusion relations are

$$S_k = \sum_{i=0}^{n} \binom{i}{k} P_{[i]}, \quad 0 \leqq k \leqq n \tag{3}$$

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and its equivalent

$$P_{[k]} = \sum_i (-1)^{i+k} \binom{i}{k} S_i, \quad 0 \leqq k \leqq n \tag{4}$$

(see Feller [2], p. 96 or Parzen [4], p. 76).

Since $P_{[k]} \geqq 0$, eq. (4) yields a system of $n+1$ inequalities for S_i:

$$0 \leqq \sum_i (-1)^{i+k} \binom{i}{k} S_i, \quad 0 \leqq k \leqq n. \tag{5}$$

Conversely, if a sequence $S_0, \ldots, S_n$ satisfies the system (5), then we may *define* $P_{[k]}$ by (4) and it is an easy matter to construct a (finite) space X, a measure μ, and a system of subsets $A_1, \ldots, A_n$ of X for which (1) and (2) hold. Thus the system (5) characterizes the possible sequences S_i arising from (1).

In what follows, a sequence is understood to be a finite sequence of length n or $n+1$ depending on whether we have a 0-th term. We use $S_0, \ldots, S_n$ as a generic sequence satisfying the system (5).

The system (5), for any fixed n, of course implies other inequalities among the S_i which give information on the sizes of intersections of sets. M. Fréchet ([3], pp. 53—75) gives several such inequalities. For example, Boole's inequality $(S_n \geqq S_1 - (n-1)S_0)$ shows (for sets) that if $\sum_{i=1}^{n} \mu(A_i) > (n-1)\mu(X)$ then $\mu(A_1 \cap \ldots \cap A_n) > 0$. Other examples include Bonferroni's inequalities and Fréchet's generalization of these, Gumbel's inequalities (generalizing Boole's) and other inequalities due to Fréchet (see below).

In the present paper we give further inequalities and new derivations of these results. In section 1, we work algebraically with (5) to derive generalizations of the above results. In section 2, we proceed geometrically using (3) to obtain other inequalities and asymptotic results as $n \to \infty$.

1. M-sequences

In order to investigate the system (5), it is convenient to rescale the sequence $S_0, \ldots, S_n$ and the associated sequence $P_{[0]}, \ldots, P_{[n]}$ defined by (4). Define (cf. Fréchet [3], p. 64)

$$M_k = S_k \Big/ \binom{n}{k}, \quad k = 0, \ldots, n; \tag{6}$$

$$R_k = P_{[k]} \Big/ \binom{n}{k}, \quad k = 0, \ldots, n. \tag{7}$$

If $A_1, \ldots, A_n$ are given sets, call any set of the type $A_{i_1} \cap \ldots \cap A_{i_k}$ $(i_1 < \ldots < i_k)$ a *k-intersection*. By convention, the only 0-intersection is X. Similarly, a *k-atom* is a set of the type $A_{i_1} \cap \ldots \cap A_{i_k} \cap \bar{A}_{j_1} \cap \ldots \cap \bar{A}_{j_{n-k}}$ where $i_1, \ldots, i_k, j_1, \ldots, j_{n-k}$ is a rearrangement of $1, 2, \ldots, n$. Thus, using eqs. (1) and (6), M_k *is the average mea-*

sure of a k-intersection. Similarly, R_k is the average measure of a k-atom. We shall use $M_0, \ldots, M_n$ as a generic sequence defined by (6) where, by our convention, the S_i satisfy (5).

1. Definition. An *S-sequence* is a sequence $S_0, \ldots, S_n$ which satisfies the inequalities (5). An *M-sequence* is a sequence $M_0, \ldots, M_n$ such that the corresponding sequence S_i defined by (6) is an S-sequence.

Remark. An S-sequence may be prolonged into a larger S-sequence. For example, using sets A_i as in (1), simply adjoin null sets to the system A_i. This is not true for an M-sequence, since eq. (6) explicitly uses n in its definition. The adjunction of null sets will, for example, reduce M_1, which is the average size of the A_i. On the other hand, we shall see that we may truncate an M-sequence by omitting its first or its last terms to obtain a new M-sequence. It turns out to be convenient to use both S- and M-sequences. Certain results will often look simpler depending on whether they are expressed in terms of the M_i or the S_i.

If we use (6) and (7) and substitute into (3) and (4), we obtain by a simple calculation

$$M_k = \sum_{i=0}^{n} \binom{n-k}{n-i} R_i, \quad 0 \leqq k \leqq n; \tag{8}$$

$$R_k = \sum_{k=0}^{n} (-1)^{i+k} \binom{n-k}{n-i} M_i, \quad 0 \leqq k \leqq n. \tag{9}$$

This suggests reversing the order of the M_i and the R_i.

If a_i is any sequence $(0 \leqq i \leqq n)$, let us write

$$\bar{a}_i = a_{n-1}, \quad 0 \leqq i \leqq n$$

for the reversed sequence. It is now an easy matter to verify that (9) becomes

$$\bar{R}_k = \Delta^k \bar{M}_i|_{i=0}, \quad k = 0, 1, \ldots, n. \tag{10}$$

(In terms of the original M_i and R_i, R_k is the k-th backward difference of M_i at $i=n$.) Since (10) is equivalent to (3), we have

2. Corollary. *If $M_0, \ldots, M_n$ is an M-sequence, then*

$$\Delta^k \bar{M}_i \geqq 0 \quad \textit{for} \quad 0 \leqq i+k \leqq n \tag{11}$$

Equivalently, in terms of M_i,

$$(-1)^k \Delta^k M_i \geqq 0 \quad \textit{for} \quad 0 \leqq i+k \leqq n. \;\blacksquare \tag{12}$$

Thus, the graph of $y=M_k$ is nonnegative, decreasing, and convex up.* This is a consequence of (12) for $k=0, 1$ and 2. This yields

* Thanks to W. R. Emerson for his suggestion that (12) holds for all $k \leqq n$.

3. Corollary. *If $M_0, \ldots, M_n$ is an M-sequence,*

(13) $$M_i \geqq 0, \quad 0 \leqq i \leqq n$$

(14) $$M_i \geqq M_j, \quad 0 \leqq i < j < n$$

(15) $$(k-i)M_j \leqq (k-j)M_i + (j-i)M_k, \quad 0 \leqq i < j < k \leqq n. \quad \blacksquare$$

Remark. (14) is due to Fréchet. (15) is due to Kai Lai Chung [1], and is a generalization of Gumbel's inequality which is equivalent to (15) for the case $i=0$ and $k=n$. We can illustrate (15) with a simple example.

4. Example. Suppose $\mu(X)=1$ and $A_1, \ldots, A_n$ $(n \geqq 11)$ are measurable subsets such that $\mu(A_i \cap A_j) \geqq 0.82$. Then $M_{11} \geqq 0.01$, and hence some 11 of the sets A_i meet in a set of measure $\geqq 0.01$.

For, we have $M_0 = 1$, $M_2 \geqq 0.82$. Thus, by (15),

(16) $$11M_2 \leqq 9M_0 + 2M_{11}$$

$$M_{11} \geqq \frac{1}{2}(11M_2 - 9M_0) \geqq 0.01.$$

The one inequality (16) becomes different inequalities in the S_i depending on the value of n, since the S_i are normalized differently for different n. Thus (16) becomes the series of inequalities

$$\frac{11S_2}{\binom{n}{2}} \leqq 9S_0 + \frac{2S_{11}}{\binom{n}{11}} \quad (n \geqq 11).$$

We shall see later that the lower bound 0.01 for M_{11} can be improved when $n > 11$. Example 4 easily generalizes.

5. Corollary. *Suppose $M_0, \ldots, M_n$ is an M-sequence, and that for some k, N with $0 < k < N \leqq n$ we have*

$$M_k > \frac{N-k}{N} M_0.$$

Then

$$M_N \geqq \frac{1}{k}\left(NM_k - (N-k)M_0\right). \quad \blacksquare$$

We can easily generalize inequalities (14) and (15). For fixed k, we have $\Delta^k \overline{M}_i \geqq 0$. But this implies that the k-th divided difference of $\overline{M}_i$ is non-negative. Thus we have

6. Corollary. *Let $M_0, \ldots, M_n$ be an M-sequence. Let $0 \leqq i_0 < \ldots < i_k \leqq n$. Define $d_\nu = \prod_{j=0, j \neq \nu}^{k} |i_j - i_\nu|$. Then*

(17) $$\sum_{\nu=0}^{k} (-1)^\nu \frac{M_{i_\nu}}{d_\nu} \geqq 0. \quad \blacksquare$$

Remark. Eq. (17) is the straightforward generalization of eqs. (13), (14) and (15). Since it uses only $\Delta^k M_i \geqq 0$ for fixed k, it does not give the complete picture for the relationship between $M_{i_0}, \ldots, M_{i_k}$.

We can use (17) to find 3-term inequalities involving M_0, M_1, M_2 for an M-sequence $M_0, \ldots, M_n$ ($n \geqq 2$). For, choose N so that $2 < N \leqq n$. Applying Corollary 6 to the sequence $0<1<2<N$, we have

$$\frac{M_0}{2N} - \frac{M_1}{N-1} + \frac{M_2}{2(N-2)} - \frac{M_N}{d_3} \geqq 0.$$

Thus

$$\frac{M_0}{N} - \frac{2M_1}{N-1} + \frac{M_2}{N-2} \geqq 0 \quad \text{for} \quad N = 3, 4, \ldots, n.$$

This is a sequence of $n-2$ inequalities on M_0, M_1, M_2, all based upon $\Delta^3 M_i \leqq 0$.

We end this section by considering the Fréchet—Bonferroni inequalities. In terms of the S_i, it states that the "tail" in the expansion of $P_{[k]}$ in (4) has the sign of the leading term:

$$\sum_{i=t}^{n} (-1)^{i+t} \binom{i}{k} S_i \geqq 0 \quad \text{for} \quad k \leqq t \leqq n.$$

Converting to the M-sequence $M_0, \ldots, M_n$, this is equivalent to

$$\sum_{i=0}^{t} (-1)^{t+i} \binom{k}{i} \overline{M}_i \geqq 0, \quad 0 \leqq t \leqq k.$$

But this follows easily from (11).

2. The Geometry of M-Sequences

For fixed n, the S-sequences are clearly closed under addition and multiplication by nonnegative numbers. Thus, the set of such sequences forms a cone in $\mathbf{R}^{n+1}$. Similarly for M-sequences $M_0, \ldots, M_n$. It is natural and convenient to normalize by taking $S_0 = M_0 = 1$. The set of normalized M-sequences is clearly a closed convex set, contained in the unit cube. For normalized S_i, we have $0 \leqq S_i \leqq \binom{n}{i}$.

7. Definition. The set $D_n \subseteqq \mathbf{R}^n$ is the convex set of all $\alpha = (M_1, \ldots, M_n) \in \mathbf{R}^n$ such that $1, M_1, \ldots, M_n$ is an M-sequence. The set $E_n \subseteqq \mathbf{R}^n$ is the convex set of all $\beta = (S_1, \ldots, S_n) \in \mathbf{R}^n$ such that $1, S_1, \ldots, S_n$ is an S-sequence.

Let $T\colon \mathbf{R}^n \to \mathbf{R}^n$ be defined by

$$T(x_1, \ldots, x_n) = \left(\binom{n}{1} x_1, \binom{n}{2} x_2, \ldots, \binom{n}{n} x_n \right).$$

Then by definition, we have

$$E_n = TD_n; \quad D_n = T^{-1} E_n.$$

In what follows, in order to avoid a multiplicity of indices, we use the following conventions. Unless otherwise stated, *points* are understood to be in $\mathbf{R}^n$. The index i is chosen so that $0 \leqq i \leqq n$. We use ν to designate as an integer $1 \leqq \nu \leqq n$, and if ϱ is a point we take $\varrho(\nu)$ as the ν-th coordinate of ϱ.

8. Theorem. *For* $0 \leqq i \leqq n$ *let the point* σ_i *be the point defined by*

$$\sigma_i(\nu) = \binom{i}{\nu}, \quad \nu = 1, \ldots, n. \tag{18}$$

Then E_n *is the convex span of the points* $\sigma_0, \sigma_1, \ldots, \sigma_n$.

Remark. $\sigma_0 = (0, 0, \ldots, 0)$, $\sigma_1 = (1, 0, \ldots, 0)$.

Proof. We go to (3). If $S_0 = 1$, we have, setting $k = 0$ in (3)

$$1 = P_{[0]} + \cdots + P_{[n]}, \quad P_{[i]} \geqq 0. \tag{19}$$

The remaining equations $(k = 1, \ldots, n)$ are simply

$$(S_1, \ldots, S_n) = \sum_{i=0}^{n} P_{[i]} \sigma_i. \tag{20}$$

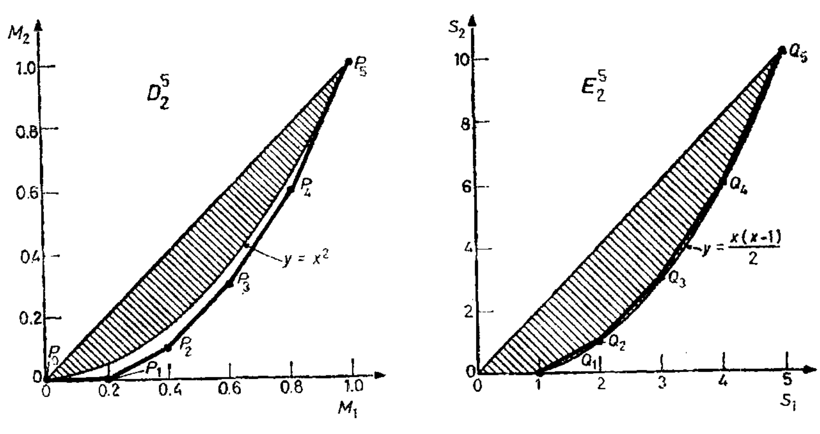
Thus, E_n is in the convex closure of $\sigma_0, \ldots, \sigma_n$.

Now if $P_{[i]}$ satisfy (19), but are otherwise arbitrary, we may use (20) to define S_ν $(1 \leqq \nu \leqq n)$. Then $S_0, S_1, \ldots, S_n$ (with $S_0 = 1$) satisfy eqs. (3) and its inverse (4), hence (5). Thus, $(S_1, \ldots, S_n)$ is in E_n, and E_n is the convex closure of $\sigma_0, \ldots, \sigma_n$. ∎

Remark. By (18) the points $\varrho_i = T^{-1} \sigma_i$ span D_n. Explicitly,

$$\varrho_i(\nu) = \frac{\sigma_i(\nu)}{\binom{n}{\nu}} = \frac{i(i-1) \ldots (i-\nu+1)}{n(n-1) \ldots (n-\nu+1)}. \tag{21}$$

If we explicitly want to indicate the dimension n, we write $\varrho_i = \varrho_i^n$ and $\sigma_i = \sigma_i^n$.



Suppose we wish to find all possible pairs (M_1, M_2) in a normalized M-sequence $1, M_1, M_2, \ldots, M_5$, or the possible pairs S_1, S_2 in a normalized S-sequence with $n=5$. We merely project D_5 (or E_5) onto $\mathbf{R}^2$ to obtain the set D_2^5 (or E_2^5) using the map $(x_1, x_2, x_3, x_4, x_5)\mapsto(x_1, x_2)$. Let P_i be the projection of ϱ_i, and Q_i be the projection of σ_i. Thus, for example,

$$Q_i = \left(i, \frac{i(i-1)}{2}\right), \quad (i = 0, \ldots, 5).$$

Clearly, the Q_i span E_2^5, since the σ_i span E_5. By considering the supporting line Q_1Q_2 we may easily obtain $S_2 \geqq S_1 - 1$ (Bonferroni's identity). Similarly, the support line Q_2Q_3 yields $S_2 \geqq 2S_1 - S_3$, etc. We shall generalize this situation below.

The Q_i's all pass through the curve $y=\binom{x}{2}$ and the P_i's all lie under the curve $y=x^2$. We now analyze this below to get upper and lower bounds for the sets D_n and E_n.

9. Definition. Let $\gamma=\gamma(t)=\gamma^n(t)$ $(0\leqq t\leqq 1)$ be the curve in $\mathbf{R}^n$ whose ν-th coordinate is

(22) $$\gamma\colon\ x_\nu = t^\nu, \quad \nu = 1, \ldots, n; \quad 0 \leqq t \leqq 1.$$

Let $\delta=\delta(s)=\delta^n(s)$ $(0\leqq s\leqq n)$ be the curve in $\mathbf{R}^n$ whose ν-th coordinate $y_\nu(s)$ is

(23) $$\delta\colon\ y_\nu(s) = \begin{cases} \binom{s}{\nu}, & \nu-1 \leqq s \leqq n \\ 0, & 0 \leqq s \leqq \nu-1. \end{cases}$$

We shall use the convention that $\binom{s}{\nu}=0$ for real s satisfying $0\leqq s\leqq \nu-1$. Thus we can simply write $y_\nu(s)=\binom{s}{\nu}$, $0\leqq s\leqq n$. We have agreement for integer $s\geqq 0$, and we now also have $\binom{s}{\nu}\geqq 0$ for $0\leqq s\leqq n$. We use Cl A to denote the convex closure of a set A.

10. Theorem. $\mathrm{Cl}\,\gamma\subseteqq D_n$; $E_n\subseteqq \mathrm{Cl}\,\delta$.

Proof. For $0\leqq t\leqq 1$, the (infinite) sequence $\alpha=1, t, t^2, \ldots, t^n, \ldots$ satisfies $-\Delta\alpha = (1-t)\alpha$. Thus, $(-1)^k\Delta^k\alpha=(1-t)^k\alpha\geqq 0$. Hence $1, t, \ldots, t^n$ is a normalized M-sequence, and $(t, t^2, \ldots, t^n)\in D_n$. This is simply $\gamma(t)\subseteqq D_n$. Taking closures we get the first inclusion.

By Theorem 8, $\sigma_0, \ldots, \sigma_n$ span E_n. But $\sigma_i(\nu)=\binom{i}{\nu}=y_\nu(i)$ by (18) and (23). Thus,

(24) $$\sigma_i = \delta(i)$$

and $\sigma_i\in\delta$. Since the σ_i span E_n, we get the second inclusion by taking closures. ∎

We now project $(x_1, \dots, x_n) \mapsto (x_1, x_N)$ to find the image of E_n in the plane. Using supporting lines we easily obtain:

11. Corollary. *Let* $1 = S_0, S_1, \dots, S_n$ *be an S-sequence. Let* $1 < N \leqq n$ *and* $S_1 \geqq N-1$. *Then if* $\varrho = [S_1]$, *we have*

$$S_N \geqq (\varrho + 1 - S_1)\binom{\varrho}{N} + (S_1 - \varrho)\binom{\varrho+1}{N}, \tag{25}$$

$$S_N \geqq \binom{S_1}{N}, \tag{26}$$

$$S_N - \binom{\nu}{N-1} S_1 \geqq \binom{\nu}{N} - \nu\binom{\nu}{N-1}, \quad (\nu = 0, 1, \dots). \ ∎ \tag{27}$$

Inequality (26) is strict unless S_1 is an integer.
Inequality (25) cannot be improved; i.e. equality is always possible.

Remark. Fréchet [3] gives (26) for $N=2$ by a different method. (27) becomes Bonferroni's identity $S_2 - S_1 + S_0 \geqq 0$ for $N=2, \nu=1$.

Similarly, by projecting $(x_1, \dots, x_n) \mapsto (x_k, x_N)$ we obtain:

12. Corollary. *Let* $S_0, \dots, S_n$ *be a normalized S-sequence and let* $1 \leqq k < N \leqq n$. *Then, for* $\nu = 0, 1, 2, \dots$, *we have*

$$\binom{\nu}{k-1} S_N \geqq \binom{\nu}{N-1} S_k + \binom{\nu}{N}\binom{\nu}{k-1} - \binom{\nu}{N-1}\binom{\nu}{k}. \tag{28}$$

Equality in (28) is attainable if $\binom{\nu}{k} \leqq S_k \leqq \binom{\nu+1}{k}$. Otherwise this is strict inequality.

For example, in Example 4 we had $M_2 \geqq 0.82$ and we showed (independently of $n \geqq 11$) that $M_{11} \geqq 0.01$. We can get a better lower bound for M_{11} depending on n. For example, if $n=13$, we have $S_2 = \binom{13}{2} M_2 \geqq 63.96$. Choosing $S_2 = 63.96$, we have $\binom{11}{2} \leqq S_2 \leqq \binom{12}{2}$. Inequality (28) for $\nu = 11$ shows that $S_{11} \geqq 9.96$ and hence $M_{11} \geqq \frac{83}{650} = 0.1276\dots$, with equality possible for $n=13$. In general, as we see below, the lower bound b_n for M_{11} will increase with n, for fixed M_2, and $b_n \to (M_2)^{11/2}$. Thus, regardless of the size of n, we will have $b_n < (0.82)^{11/2} = 0.335\dots$. We now consider such asymptotic results.

13. Lemma. *Let* $1 \leqq N \leqq n$ *and let* $(M_1, \dots, M_n) \in D_n$. *Then if* $M_1 \geqq \frac{N-1}{n}$, *we have*

$$M_N \geqq M_1^N - \frac{M_1^{N-1}(1-M_1)N^2}{2(n-N)} \geqq M_1^N - \frac{N}{4(n-N)}. \tag{29}$$

Proof. By Corollary 11, $S_N \geqq \binom{S_1}{N}$. But $S_N = \binom{n}{N} M_N$ and $S_1 = nM_1$. Thus,

$$\binom{n}{N} M_N \geqq \binom{nM_1}{N},$$

$$M_N \geqq \frac{nM_1(nM_1-1)\dots(nM_1-N+1)}{n(n-1)\dots(n-N+1)},$$

if $nM_1 \geqq N-1$. Now set

$$f_N(x) = \frac{x\left(x-\frac{1}{n}\right)\dots\left(x-\frac{N-1}{n}\right)}{\left(1-\frac{1}{n}\right)\dots\left(1-\frac{N-1}{n}\right)}, \quad \left(x \geqq \frac{N-1}{n}\right). \tag{30}$$

It suffices to show that

$$f_N(x) \geqq x^N - \frac{x^{N-1}(1-x)N^2}{2(n-N)} \geqq x^N - \frac{N}{4(n-N)}. \tag{31}$$

We estimate $f_N(x)$ by noting that

$$\frac{x-\frac{i}{n}}{1-\frac{i}{n}} = x - \frac{i(1-x)}{n-i}.$$

Thus,

$$f_N(x) = \prod_{i=0}^{N-1}\left(x - \frac{i(1-x)}{n-i}\right) \geqq x^N - x^{N-1}(1-x)\sum_{i=0}^{N-1}\frac{i}{n-i}$$

by induction on N. But

$$\sum_{i=0}^{N-1}\frac{i}{n-i} \leqq \int_0^N \frac{t\,dt}{n-t} = -n\log\left(1-\frac{N}{n}\right) - N \leqq \frac{N^2}{2(n-N)}$$

using a simple power series estimate for $-\log(1-x)$. This proves the first of the inequalities (31). Using calculus, we have $x^{N-1}(1-x) \leqq 1/2N$ $(0 \leqq x \leqq 1)$. This is the rest of the inequality. ∎

Remark. Thus, for any $\varepsilon > 0$, if $n \geqq N\left(1+\frac{1}{4\varepsilon}\right)$, and $n \geqq (N-1)/M_1$ we have $M_N \geqq$ $\geqq M_1^N - \varepsilon$.

14. Theorem. *Let* $N \leqq n$ *and let* $\Pi_N^n\colon \mathbf{R}^n \to \mathbf{R}^N$ *be the projection map given by* $\Pi_N^n(x_1, \dots, x_n) = (x_1, \dots, x_N)$. *Let* $D_N^n = \Pi_N^n D_n$. *Then the sequence* D_N^n *is a strictly* *monotonic decreasing sequence of sets with*

$$\bigcap_{n \geqq N} D_N^n = \mathrm{Cl}\,\gamma^N. \tag{32}$$

Proof. If $M_0, \ldots, M_{n+1}$ is an M-sequence, then so is $M_0, \ldots, M_n$. Thus, the sets D_N^n decrease. The point $\alpha=(1/n, 0, \ldots, 0)\in \mathbf{R}^N$ is in D_N^n since $\alpha=\Pi_N^n \varrho_1^n$. But $\alpha \notin D_N^{n+1}$ since the $(n+1)$-st difference of $0, \ldots, 0, 1/n, 1$ is negative. Thus D_N^n strictly decrease.

Since D_n is spanned by $\varrho_0, \ldots, \varrho_n$, D_N^n is spanned by $\Pi_N^n \varrho_i = P_i \in \mathbf{R}^N$. By (21), the k-th coordinate of P_i is $f_k\left(\frac{i}{n}\right)$, where $f_k=f_{k,n}$ is the function defined in (30) for $0 \leqq x \leqq 1$. But by (31), $f_k(x)\to x^k$ uniformly for $0\leqq x \leqq 1$ as $n\to\infty$ for $k=1, \ldots, N$. Thus, for any $\varepsilon>0$, $\left|P_i-\gamma^N\left(\frac{i}{n}\right)\right|<\varepsilon$ for sufficiently large n and $0\leqq i\leqq n$. Thus, all generators of D_N^n are within ε of the curve γ^N for n sufficiently large. This proves (32). ▮

Remark. The estimate (31) shows that $n=O\left(\frac{1}{\varepsilon}\right)$ is sufficient to insure that all points of D_N^n are within ε of $\mathrm{Cl}\,\gamma^N$.

We use Theorem 14 by giving some asymptotic inequalities for M-sequences.

15. Definition. If $f(t)=a_0+a_1t+\ldots+a_Nt^N$ is a polynomial of degree $\leqq N$, we write $l_f(x)=a_0+a_1x_1+\ldots+a_Nx_n$ for $x=(x_1, \ldots, x_N)\in\mathbf{R}^N$.

Thus, by definition of $\gamma^N(t)$, we have

$$l_f(\gamma^N(t)) = f(t). \tag{33}$$

16. Corollary. *Let $f(t)$ be a polynomial of degree $\leqq N$. Then D_N^n is contained in the half space $l_f(x)\geqq 0$ for sufficiently large n if and only if $f(t)>0$ for $0<t<1$.*

Proof. *Necessity.* By the proof of Theorem 10, if $0<t_0<1$, all differences of the sequences $t_0^N, t_0^{N-1}, \ldots, t_0, 1$ are strictly positive, and so $\gamma(t_0)$ is in the interior of D_N^n. Thus, if D_N^n is contained in the half plane $l_f(x)\geqq 0$, we have $l_f(\gamma(t_0))>0$ or by (32), $f(t_0)>0$ for $0<t_0<1$.

Sufficiency. We first observe that if D_N^n is contained in the half space $l_f(x)\geqq 0$ then D_N^{n+1} is contained in the half spaces $l_g(x)\geqq 0$ and $l_h(x)\geqq 0$ where $g(t)=tf(t)$ and $h(t)=(1-t)f(t)$. This is a simple consequence of the fact that if $(M_0, \ldots, M_{n+1})$ is an M-sequence, then so are $(M_1, \ldots, M_{n+1})$ and $(-\Delta M_0, \ldots, -\Delta M_n)$. Now suppose $f(t)>0$ for $0<t<1$. By factoring out powers of t and $(1-t)$, we have $f(t)= =t^{N_1}g(t)(1-t)^{N_2}$ where $g(t)>0$ for $0\leqq t\leqq 1$. Suppose $g(t)\geqq\varepsilon>0$ in this interval. Thus, $l_g(\gamma^N(t))=g(t)\geqq\varepsilon$ by (33) and, by Theorem 14, D_N^n is in the half space $l_g(x)\geqq 0$ for sufficiently large n. By the above observations, D_N^n is in the half space $l_f(x)\geqq 0$ for large n, since $f(t)=t^{N_1}g(t)(1-t)^{N_2}$. This completes the proof. ▮

Remark. Corollary 16 gives all possible linear inequalities on a fixed number of terms $M_0, \ldots, M_N$ which are valid for all M-sequences of sufficiently large length.

For example, $9M_2+M_0 \geqq 5M_3+5M_1$ for all M-sequences of sufficiently large length, since $-5t^3+9t^2-5t+1$ satisfies the hypothesis of this corollary. This inequality is not true, for example, for the M-sequence 1, 1/3, 0, 0.

Finally, by projecting $(x_1, \ldots, x_n) \mapsto (x_k, x_N)$, Theorem 14 gives

17. Corollary. *Let* $1 < k \leqq N$ *and* $\varepsilon > 0$. *Then for n sufficiently large,*

$$M_N > M_k^{N/k} - \varepsilon. \quad \blacksquare$$

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